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THE EMISSION OF A GAS JET FROM A CONICAL NOZZLE†

L. M. ZIGANGAREYEVA and O. M. KISELEV

Kazan

e-mail: kiselevoleg@hotmail.com

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The axially symmetric, subsonic emission of a jet of compressible fluid from a conical nozzle is considered. Subject to the assumptions that a solution of the corresponding boundary-value problem exists and that a certain asymptotic expansion holds in the case of this solution, it is proved that, in an axially symmetric jet of a compressible fluid at a critical pressure on the free surface, the gas velocity reaches the sound velocity in a certain plane which is perpendicular to the axis and located at a finite distance from the nozzle edge. Results are presented for a jet of a perfect gas with an adiabatic exponent $\gamma = 1.4$. Approximate formulae are given which enable one to determine the form of a jet with a sonic velocity on the free surface. © 2000 Elsevier Science Ltd. All rights reserved.

The problem of the emission of an axially symmetric jet of an incompressible fluid from a funnel-shaped nozzle has been considered by a number of authors. However, the results obtained (see, [1–3]) were not of a high accuracy. An effective method for solving the problem of the emission of an axially symmetric jet of a compressible fluid (a gas), based on the use of the variables of a velocity hodograph, has been proposed in [4]. However, in [4] and in later papers, there are no results of calculations for the subsonic emission of gas and no method is given for the calculation of a jet with a sonic velocity on the free boundary.

A development of the method in [4], which also enables one, in particular, to use it in the case of a sonic velocity on the free boundary, is given below.

1. FORMULATION OF THE PROBLEM

Consider an axially symmetric, subsonic emission of a jet of an ideal, compressible fluid from a semi-infinite conical nozzle. We shall assume that there are no external forces and that the jet is a steady, barotropic, irrotational flow. In the half-plane of the cylindrical coordinates x and r, the flow domain is bounded by the x axis, the generatrix of the cone $a_1 b$, which makes an angle θ_0 with the x axis, and the free surface bc. The r axis passes through the edge of the nozzle b (Fig. 1a).

Suppose V and ρ are the velocity and the density of the fluid, θ is the angle of inclination of the velocity vector to the x axis, M is the Mach number, V_c, ρ_c, M_c are the values of V, ρ and M at the free surface ($M_c \leqslant 1$), $\tau = V/V_c$, $\nu = \rho/\rho_c$, $Y = \nu\tau r^2/2$ and ψ is the stream function, introduced using the relations

$$\tau \cos \theta = (r\nu)^{-1}\psi_r,\ \tau \sin \theta = -(r\nu)^{-1}\psi_x,$$

(subscripts are used to denote partial derivatives).

The rectangle τ, θ corresponds to the flow domain ina the plane of the variables τ and θ (Fig. 1b; the segment $\Sigma = \{(\tau, \theta)\,|\,0 < \tau < 1, -\theta_0 < \theta < 0\}$ corresponds to an infinitely distant stagnation point of the flow and the points B and C correspond to the points b and c).

It is known [4–6] that the functions $\psi(\tau, \theta)$, $r(\tau, \theta)$, $x(\tau, \theta)$ satisfy the relations

$$\begin{aligned} &R = R(\psi, Y) = \sin\theta S^2 L - P_\theta S + P S_\theta = 0 \\ &L = L(\psi) = (1 - M^2)\psi_{\theta\theta} + \tau^2\psi_{\tau\tau} + (1 + M^2)\tau\psi_\tau \\ &P = P(\psi) = \sin^2\theta(\tau^2\psi_\tau^2 + (1 - M^2)\psi_\theta^2) \end{aligned} \quad (1.1)$$

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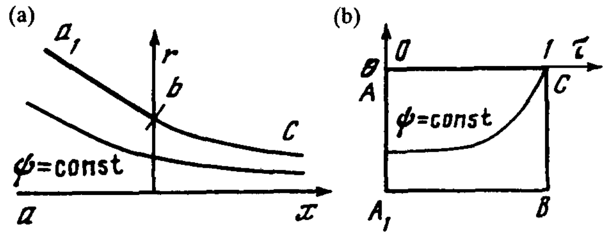

Fig. 1.

$$S = S(\psi, Y) = 2Y + \psi_\theta \sin\theta$$

$$\begin{Bmatrix} x_\tau \\ r_\tau \end{Bmatrix} = \frac{1}{r\nu\tau^2}\left((M^2-1)\psi_\theta \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} + \frac{P}{S}\begin{Bmatrix} \operatorname{ctg}\theta \\ 1 \end{Bmatrix} \mp \tau\psi_\tau \begin{Bmatrix} \sin\theta \\ \cos\theta \end{Bmatrix}\right), \tag{1.2}$$

$$\begin{Bmatrix} x_\theta \\ r_\theta \end{Bmatrix} = \frac{1}{r\nu\tau}\left(\tau\psi_\tau \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} \mp \psi_\theta \begin{Bmatrix} \sin\theta \\ \cos\theta \end{Bmatrix}\right)$$

Without loss in generality, it may be assumed that

$$\psi = 0 \text{ on } AC,\ \psi = 1 \text{ on } A_1BC \text{ and } \psi = (1-\cos\theta)/(1-\cos\theta_0) \text{ on } AA_1 \tag{1.3}$$

(the last of conditions (1.3) holds for the whole of the radial flow domain towards the sink).

Using relations (1.2), we express Y in terms of ψ:

$$Y = Y(\psi) = \psi\cos\theta + \int_0^\theta (\tau\psi_\tau + \psi)\sin\theta d\theta \tag{1.4}$$

Substituting expressions (1.4) into Eqs (1.1) ($S(\psi, Y(\psi)) = S(\psi)$, $R(\psi, Y(\psi)) = R(\psi)$), we obtain an integro-differential equation in ψ. Relations (1.1), (1.3) and (1.4) define a boundary-value problem for ψ in he domain Σ. The solution of this problem will be sought in the form $\psi = \psi^0 + \chi$, where ψ^0 is he leading part of the asymptotic expansion of the stream function in the neighbourhood of the singular point C and χ is a smoother function which is found by the method of finite differences.

2. ASYMPTOTIC EXPANSION OF Ψ WHEN $M_C < 1$

We shall assume that M and v are known functions of τ, which are analytic in the neighbourhood of the point $\psi = 1$. In this case, the coefficients, which depend on τ, in expressions (1.1) and (1.2) can be expanded in power series in $\zeta = \tau - 1$

$$\tau^2 = 1 + 2\zeta + \zeta^2,\quad 1 - M^2 = \sum_{k=0}^{\infty} u_k\zeta^k,\quad \tau(1+M^2) = \sum_{k=0}^{\infty} q_k\zeta^k$$

$$u_0 = 1 - M_c^2,\quad u_k = -\frac{1}{k!}\frac{d^kM^2}{d\tau^k}\bigg|_{\tau=1},\quad k = 1,2,\ldots\ (u_1 < 0) \tag{2.1}$$

$$q_0 = 2 - u_0,\quad q_1 = q_0 - u_1,\quad q_k = -u_{k-1} - u_k,\quad k = 2,3,\ldots$$

$$\nu^{-1} = 1 + \zeta M_c^2 - \frac{1}{2}\zeta^2(u_0M_c^2 + u_1) + \ldots$$

Putting $M_c < 1$, we introduce the variables σ and ω

$$\sigma = (\theta^2 + \alpha^2\zeta^2)^{1/2},\quad \omega = \operatorname{arctg}\frac{\theta}{\alpha\zeta},\quad \alpha = u_0^{1/2} = (1 - M_c^2)^{1/2} \tag{2.2}$$

(σ and ω are the distance to the origin of the coordinate system and the central angle in the plane of the variables $\alpha\zeta$, and θ; $\omega = -\pi$ on AC and $\omega = -\pi/2$ on CB). According to (2.2)

$$\zeta = \alpha^{-1}\sigma\cos\omega, \quad \theta = \sigma\sin\omega \tag{2.3}$$

$$\sigma_\theta = \sin\omega, \quad \sigma_\tau = \alpha\cos\omega, \quad \omega_\theta = \sigma^{-1}\cos\omega, \quad \omega_\tau = -\alpha\sigma^{-1}\sin\omega \tag{2.4}$$

System (1.1), (1.4) can be written in the form

$$L(\psi) = N(\psi), \quad N(\psi) = (SP_\theta - PS_\theta)(S^2\sin\theta)^{-1} \tag{2.5}$$

Suppose $Q(\psi) = L(\psi) - N(\psi)$ and ψ_0 is a function which satisfies the equation $Q(\psi) \to 0$ when (τ, θ) B Σ, $(\tau, \theta) \to (1,0)$ as well as the conditions $\psi_0 = 0$ on AC and $\psi_0 = 1$ on CB. We shall seek ψ_0 in the form of an asymptotic expansion in the small parameter σ, putting

$$\begin{gathered} \psi_0 = \psi_1 + \psi_2 + \ldots, \qquad \psi_k = h_k(\sigma)f_k(\omega) \\ h_{k+1}(\sigma)/h_k(\sigma) \to 0 \quad \text{при} \quad \sigma \to 0, \quad k = 1,2,\ldots \end{gathered} \tag{2.6}$$

and requiring that the following conditions be satisfied

$$\begin{gathered} \psi_k = 0 \text{ при } \omega = -\pi, \quad k = 1,2,\ldots \\ \psi_1 = 1 \text{ при } \omega = -\pi/2, \quad \psi_k = 0 \text{ при } \omega = -\pi/2,\ k = 2,3,\ldots \end{gathered} \tag{2.7}$$

Suppose ψ is the solution of boundary-value problem (1.1), (1.3), (1.4). The asymptotic expansion of ψ in the small parameter σ is obviously identical to expansion (2.6) as long as the functions are uniquely defined.

It is natural to seek the leading term in expansion (2.6) in the form $\psi_1 = f_1(\omega)$. Using relations (2.1), (2.3) and (2.4), it can be shown that, here,

$$\begin{aligned} & L(\omega_1) = L_1 + \Delta L_1, \quad P(\psi_1) = P_1 + \Delta P_1, \quad S(\psi_1) = S_1 + \Delta S_1, \quad R(\psi_1) = R_1 + \Delta R_1 \\ & L_1 = \alpha^2\psi_{1\theta\theta} + \psi_{1\tau\tau} = \alpha^2\sigma^{-2}f_1'', \quad \Delta L_1 = O(\sigma^{-1}) \\ & P_1 = \theta^2(\psi_{1\tau}^2 + \alpha^2\psi_{1\theta}^2) = \alpha^2\sin^2\omega f_1'^2, \quad \Delta P_1 = O(\sigma) \\ & S_1 = 2\psi_1 + \theta\psi_{1\theta} = 2f_1 + \sin\omega\cos\omega f_1', \quad \Delta S_1 = O(\sigma) \\ & P_{1\theta} = \alpha^2\sigma^{-1}\sin 2\omega(\cos\omega f_1'^2 + \sin f_1'f_1'') \\ & S_{1\theta} = \sigma^{-1}[\cos\omega(1 + 2\cos^2\omega)f_1' + \sin\omega\cos^2\omega f_1''] \\ & R_1 = \theta S_1^2 L_1 - P_{1\theta}S_1 + P_1 S_{1\theta} = O(\sigma^{-1}), \quad \Delta P_1 = O(1) \end{aligned} \tag{2.8}$$

Equating R_1, that is, the leading term in the expansion of $R(\psi_1)$ in powers of σ, to zero, we obtain the equation

$$4f_1^2 f_1'' - 4\cos^2\omega f_1 f_1'^2 + \sin\omega\cos\omega f_1'^3 = 0 \tag{2.9}$$

Taking account of (2.7), we require that the following conditions should be satisfied

$$f_1(-\pi) = 0, \quad f_1(-\pi.2) = 1 \tag{2.10}$$

A numerical–analytic investigation shows that boundary-value problem (2.9), (2.10) has a unique solution: a monotonically increasing function $f_1(\omega)$ can be obtained by numerical integration of Eq (2.9), following for the fact that the expansions hold for the ends of the interval $[-\pi, -\pi/2]$.

$$f_1(-\pi + u) = q^{-2}\left(u^2 - \frac{1}{3}u^4 + \frac{23}{180}u^6 - \frac{113}{2520}u^8 + \ldots\right), \quad q = 0{,}83166$$

$$f_1\left(-\frac{\pi}{2} - u\right) = 1 - qu - \frac{1}{24}q^3u^3 + \left(\frac{1}{12}q^2 - \frac{1}{24}q^4\right)u^4 + \left(\frac{7}{120}q^3 - \frac{27}{640}q^5\right)u^5 + \ldots$$

Note that the relation $f_1(\omega)$ has been previously given in [4] in parametric form, which is inconvenient for practical applications without any indication of the method used to determine it

$$f_1 = t^2/t_0^2, \quad \omega = -\pi + \mathrm{arctg}[J_1(2t)/J_0(2t)], \quad t \in [0, t_0]$$

where J_1, J_0 are Bessel functions and t_0 is the least root of the equation $J_0(2t) = 0$, $t_0 = q^{-1}$.

The second term of expansion (2.6) can be found in the form $\psi_2 = \sigma f_2(\omega)$. The differential equation for $f_2(\omega)$ is obtained by equating to zero the term of the order of unity in the expansion of the expression $R(\psi_1 + \psi_2)$ with respect to σ. However, the function $\psi_1 = f_1(\omega)$, which is the leading term in the expansion of the required function ψ with respect to σ is sufficient for practical purposes. Note that $Q(\psi) = O(\sigma^{n-1})$ when $R(\psi) = O(\sigma^n)$ and, consequently, $Q(\psi_1) = O(\sigma^{-1})$.

3. ASYMPTOTIC EXPANSION OF Ψ WHEN $M_c = 1$

When $M_c = 1$, the coefficient u_0 in (2.1) vanishes, which leads to a change in the type of singularity at the point C. Putting $M_c = 1$, we introduce the variables $\varkappa$ and β:

$$\varkappa = [(\mu\theta)^2 + |\zeta|^3]^{1/2}, \quad \beta = \mathrm{arctg}\frac{\mu\theta}{|\zeta|^{3/2}}, \tag{3.1}$$

($\varkappa$ and β are the distance from the origin of the coordinate system and the central angle in the lane of the transformed variables of the hodograph $\xi_1 = |\xi|^{3/2}$, $\theta_1 = \mu\theta$; $\beta = 0$ on AC and $\beta = -\pi/2$ on CB). By (3.1)

$$\zeta = -\varkappa^{2/3}(\cos\beta)^{2/3}, \quad \theta = \mu^{-1}\varkappa\sin\beta \tag{3.2}$$

$$\varkappa_\theta = \mu\sin\beta, \quad \varkappa_\tau = -\frac{3}{2}\varkappa^{1/3}(\cos\beta)^{4/3}$$

$$\beta_\theta = \mu\varkappa^{-1}\cos\beta, \quad \beta_\tau = \frac{3}{2}\varkappa^{-2/3}\sin\beta(\cos\beta)^{1/3}$$

We shall seek the function ψ_0 in the form

$$\psi_0 = \psi_1 + \psi_2 + \ldots, \quad \psi_k = d_k(\varkappa)g_k(\beta)$$
$$d_{k+1}(\varkappa)/d_k(\varkappa) \to 0 \quad \text{when} \quad \varkappa \to 0, \quad k = 1,2,\ldots \tag{3.3}$$

while requiring that the following conditions are satisfied

$$\psi_k = 0 \quad \text{when} \quad \beta = 0, \quad k = 1,2,\ldots$$
$$\psi_1 = 1 \quad \text{when} \quad \beta = -\pi/2, \quad \psi_k = 0 \quad \text{when} \quad \beta = -\pi/2, \quad k = 2,3,\ldots \tag{3.4}$$

We shall seek the leading term in expansion (3.3) in the form $\psi_1 = g_1(\beta)$. Using relations (2.1) and (3.2), it can be shown that, in representations (2.8).

$$L_1 = u_1\zeta\psi_{1\theta\theta} + \psi_{1\tau\tau} = \frac{9}{4}\varkappa^{-4/3}(\cos\beta)^{2/3}\left(-\frac{1}{3}\mathrm{tg}\beta g_1' + g_1''\right), \quad \Delta L_1 = O(\varkappa^{-2/3})$$

$$P_1 = \theta^2(\psi_{1\tau}^2 + u_1\zeta\psi_{1\theta}^2) = \frac{9}{4}\mu^{-2}\varkappa^{2/3}\sin^2\beta(\cos\beta)^{2/3}g_1'^2, \quad \Delta P_1 = O(\varkappa^{4/3})$$

$$S_1 = 2\psi_1 + \theta\psi_{1\theta} = 2g_1 + \sin\beta\cos\beta g_1', \quad \Delta S_1 = O(\varkappa^{2/3})$$

$$P_{1\theta} = \frac{9}{4}\mu^{-1}\varkappa^{-1/3}\sin 2\beta(\cos\beta)^{2/3}(\cos\beta g_1'^2 + \sin g_1'g_1'')$$

$$S_{1\theta} = \mu\varkappa^{-1}[\cos\beta(1 + 2\cos^2\beta)g_1' + \sin\beta\cos^2\beta g_1'']$$

$$R_1 = \theta S_1^2 L_1 - P_{1\theta}S_1 + P_1 S_{1\theta} = O(\varkappa^{-1/3}), \quad \Delta R_1 = O(\varkappa^{1/3})$$

Equating R_1, the leading term in the expansion of $R(\psi_1)$ in powers of $\varkappa$, to zero, we obtain equation

$$4g_1^2g_1''+\sin\beta\cos\beta\left(1-\frac{1}{3}\sin^2\beta\right)g_1'^3-\frac{4}{3}\operatorname{tg}\beta g_1^2g_1'-\left(4-\frac{8}{3}\sin^2\beta\right)g_1g_1'^2=0 \qquad (3.5)$$

Taking account of conditions (3.4), we require that the following conditions be satisfied

$$g_1(0)=0, \qquad g_1(-\pi/2)=1 \qquad (3.6)$$

Investigation shows that boundary-value problem (3.5), (3.6) has a unique solution: a monotonically decreasing function $g_1(\beta)$ can be obtained by numerical integration of Eq. (3.5), taking account of the fact that the expansions

$$g_1(\beta)=a\left(\beta^2+\frac{1}{135}\beta^6+\frac{10}{1701}\beta^8-\frac{1}{18225}\beta^{10}+\ldots\right), \quad a=0.31247 \qquad (3.7)$$

$$g_1\left(-\frac{\pi}{2}+t\right)=1-pt^{2/3}+\frac{1}{6}p^2t^{4/3}-\left(\frac{1}{72}p+\frac{1}{648}p^4\right)t^{8/3}+\left(\frac{41}{1080}p^2-\frac{2}{1215}p^5\right)t^{10/3}+\ldots,$$

$$p=1.14967$$

hold at the ends of the interval $[-\pi/2, 0]$.

We shall seek the function ψ_2 in expression (3.3) in the form $\psi_2=\varkappa^{3/2}g_2(\beta)$. Here, according to relations (2.1) and (3.2)

$$L(\psi_1+\psi_2)=L_2+\Delta L_2, \quad P(\psi_1+\psi_2)=P_2+\Delta P_2$$

$$S(\psi_1+\psi_2)=S_2+\Delta S_2, \quad R(\psi_1+\psi_2)=R_2+\Delta R_2$$

$$L_2=u_2\zeta^2\psi_{1\theta\theta}+2\zeta\psi_{1\tau\tau}+2\psi_{1\tau}+u_1\zeta\psi_{2\theta\theta}+\psi_{2\tau\tau}=$$

$$=\varkappa^{-2/3}\left\{(\cos\beta)^{1/3}\left[\left(2\delta\sin\beta\cos^2\beta-\frac{9}{2}\sin\beta+9\sin^3\beta\right)g_1'-\right.\right.$$

$$\left.\left.-\left(\delta\cos^3\beta+\frac{9}{2}\sin^2\beta\cos\beta\right)g_1''\right]+\frac{9}{4}(\cos\beta)^{2/3}\left(\frac{2}{3}g_2-\frac{1}{3}\operatorname{tg}\beta g_2'+g_2''\right)\right\}, \qquad \Delta L_2=O(1)$$

$$P_2=\theta^2[2\zeta\psi_{1\tau}^2+u_2\zeta^2\psi_{1\theta}^2+2(\psi_{1\tau}\psi_{2\tau}+u_1\zeta\psi_{1\theta}\psi_{2\theta})]=$$

$$=\mu^{-2}\varkappa^{4/3}\left[\frac{9}{2}(\cos\beta)^{2/3}\sin^2\beta g_1'g_2'-(\cos\beta)^{4/3}\left(\delta\sin^2\beta\cos^2+\frac{9}{2}\sin^4\beta\right)\right]$$

$$\Delta P_2=O(\varkappa^2)$$

$$S_2=2\psi_2+\theta\psi_{2\theta}=\varkappa^{2/3}\left[\left(2+\frac{2}{3}\sin^2\beta\right)g_2+\sin\beta\cos\beta g_2'\right], \quad \Delta S_2=O(\varkappa^{4/3})$$

$$P_{2\theta}=\mu^{-1}\varkappa^{1/3}[-(\cos\beta)^{10/3}(2\delta\sin\beta-4\delta\sin^3\beta+18\sin^3\beta)g_1'^2-$$

$$-(\cos\beta)^{7/3}(2\delta\sin^2\beta\cos^2\beta+9\sin^4\beta)g_1'g_1''+$$

$$\left.+(\cos\beta)^{2/3}(9\sin\beta-6\sin^3\beta)g_1'g_2'+\frac{9}{2}(\cos\beta)^{5/3}\sin^2\beta(g_1''g_2'+q_1'q_2'')\right]$$

$$S_{2\theta}=\mu\varkappa^{-1/3}\left[\left(\frac{8}{3}\sin\beta-\frac{8}{9}\sin^3\beta\right)g_2+\left(3\cos\beta-\frac{2}{3}\sin^2\beta\cos\beta\right)g_2'+\sin\beta\cos^2\beta g_2''\right]$$

$$R_2=\theta(2S_1S_2L_1+S_1^2L_2)-P_{1\theta}S_2-P_{2\theta}S_1+S_{1\theta}P_2+S_{2\theta}P_1=O(\varkappa^{1/3}), \quad \Delta R_2=O(\varkappa)$$

$$\delta=-\mu^2u_2=-\frac{9}{4}\frac{u_2}{|u_1|}$$

Equating the leading term in the expansion of R_2 in powers of $\varkappa$, $R(\psi_1=\psi_2)$ to zero, we obtain the equation for $g_2(\beta)$:

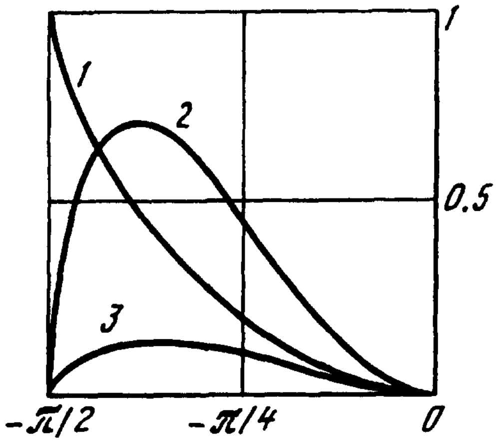

Fig. 2.

$$Eg_2'' + Fg_2' + Gg_2 = H_1 + \delta H_2$$

$$E = g_1^2$$

$$F = -\frac{1}{3}\operatorname{tg}\beta g_1^2 - (2 - \frac{2}{3}\sin^2\beta) g_1 g_1' + \sin\beta\cos\beta\left(\frac{3}{4} - \frac{1}{4}\sin^2\beta\right) g_1'^2$$

$$G = \frac{2}{3}g_1^2 - \frac{8}{9}\frac{\sin^3\beta}{\cos\beta} g_1 g_1' - \cos^2\beta\left(1 - \frac{1}{6}\sin^2\beta\right) g_1'^2 + \left(2 + \frac{2}{3}\sin^2\beta\right) g_1 g_1''$$

$$H_1 = -(\cos\beta)^{-1/3}[\sin\beta(2 - 4\cos^2\beta) g_1^2 g_1' + 2\sin^2\beta\cos\beta(g_1 g_1'^2 - g_1^2 g_1'')]$$

$$H_2 = -(\cos\beta)^{-1/3}\left[\frac{8}{9}\sin\beta\cos^2\beta g_1^2 g_1' + \frac{4}{9}\cos^3\beta(g_1 g_1'^2 - g_1^2 g_1'') - \frac{1}{9}\sin\beta\cos^4\beta g_1'^3\right]$$

We shall represent $g_2(\beta)$ in the form $g_2(\beta) = \varphi_1(\beta) + \delta\varphi_2(\beta)$ by submitting the functions φ_k to the conditions

$$E\varphi_k'' + F\varphi_k' + G\varphi_k = H_k, \quad \varphi_k(0) = \varphi_k(-\pi/2) = 0, \quad k = 1,2 \tag{3.8}$$

Analysis shows that boundary-value problems (3.8) are uniquely solvable and that the functions $\varphi_1(\beta)$, $\varphi_2(\beta)$ are non-negative and can be found by numerical integration of Eq. (3.8) when account is taken of the fact that the expansions

$$\begin{aligned}
&\varphi_1(\beta) = a\left(3\beta^3 - \beta^4 + \frac{1}{45}\beta^6 + \ldots\right)\\
&\varphi_1\left(-\frac{\pi}{2} + t\right) = 3t^{2/3} - 3pt^{4/3} + \frac{1}{2}p^2t^2 + \ldots\\
&\varphi_2(\beta) = a\left(\frac{8}{9}\beta^2 - \frac{16}{27}\beta^4 + \frac{208}{1215}\beta^6 + \ldots\right)\\
&\varphi_2\left(-\frac{\pi}{2} + t\right) = \frac{4}{9}t^{2/3} - \frac{8}{27}pt^{4/3} + \frac{2}{81}p^2t^2 + \ldots
\end{aligned} \tag{3.9}$$

hold at the ends of the interval $[-\pi/2, 0]$.

The relations $g_1(\beta)$, $\varphi_1(\beta)$, $\varphi_2(\beta)$ are shown by curves 1–3 respectively in Fig. 2.

It can be shown that, for ψ_1 and ψ_2, found when $M_c = 1$, $Q(\psi_1) = O(\varkappa^{-2/3})$. $Q(\psi_1 + \psi_2) = Q(1)$. It is obvious that the functions $\psi_1 = g_1(\beta)$ and $\psi_2 = \varkappa^{2/3}(\varphi_1(\beta) + \delta\varphi_2(\beta))$ serve as the initial terms of the expansion of the required function ψ with respect to the small parameter $\varkappa$.

4. ANALYSIS OF THE FUNCTION Ψ WHEN $M_c = 1$, $\beta \to 0, -\pi/2$

By relations (3.7) and (3.9), we have

$$g_1(\beta)+\varkappa^{2/3}(\varphi_1(\beta)+\delta\varphi_2(\beta))=\begin{cases}O(\beta^2) & \text{when } \beta\to 0\\ 1+O(t^{2/3}) & \text{when } \beta=-\pi/2+t,\ t\to 0\end{cases} \tag{4.1}$$

We shall show that, when $M_c = 1$, similar relations also hold for the required stream function $\psi = \psi(\varkappa, \beta)$.

According to the first conditions (1.3), $\psi(\varkappa, 0) = 0$. When $\beta \to 0$, we can represent $\psi(\varkappa, \beta)$ in the form

$$\psi(\varkappa,\beta)=N(\varkappa)b_1(\beta)+O(b_2(\beta)) \tag{4.2}$$

assuming that $b_1(\beta)$ is of the order of magnitude of β^ε or β^n or and that $\beta^{n\pm\varepsilon}$ is of the order of magnitude of β^m or $\beta^{m\pm\varepsilon}$, when m and n are positive constants, $m > n$ and ε is a positive quantity which may be as small as desired (according to (4.1), $n \leqslant 2$). It can be shown that $\beta b_1'/b_1 = O(\beta^\varepsilon)$ when $b_1 = O(\beta^\varepsilon)$ and, in the remaining cases, $\beta b_1'/b_1 = O(1)$ and that $\beta b_1''/b_1' = O(1)$ always.

We shall use the notation $O(\delta_1(\beta), \delta_2(\beta))$ bearing $O(\delta_0(\beta))$ in mind here, where $\delta_0(\beta)$ is that one of the functions $\delta_1(\beta)$, $\delta_2(\beta)$ which tends more slowly to zero when $\beta \to 0$. Using relations (3.2), we obtain from (4.2) that

$$\psi_\tau=O(b_1),\quad \psi_{\tau\tau}=O(b_1)$$

$$\psi_\theta=\mu N\varkappa^{-1}b_1'+O(\beta b_1,b_2'),\quad \psi_{\theta\theta}=\mu^2N\varkappa^{-2}b_1''+O(b_1,b_2'')$$

$$L=\mu^2(1-M^2)N\varkappa^{-2}b_1''+O(b_1,b_2'')$$

$$P=(1-M^2)N^2\beta^2b_1'^2+O(\beta^2b_1^2,\beta^2b_1'b_2')$$

$$S=N(2b_1+\beta b_1')+O(\beta^2b_1,b_2)$$

$$P_\theta=2\mu(1-M^2)N^2\varkappa^{-1}(\beta b_1'^2+\beta^2b_1'b_1'')+O(\beta b_1^2,\beta b_1'b_2')$$

$$S_\theta=\mu N\varkappa^{-1}(3b_1'+\beta b_1'')+O(\beta b_1,b_2')$$

$$R=R_1+\Delta R_1$$

$$R_1=\mu(1-M^2)N^3\varkappa^{-1}(4\beta b_1^2b_1''-4\beta b_1b_1'^2+\beta^2b_1'^3),\quad \Delta R_1=O(\beta b_1^3,b_1^2b_2')$$

The quantity ΔR_1, when $\beta \to 0$, is of a higher order of smallness than each of the terms appearing in R_1. It therefore follows from the equality $R(\psi(\varkappa, \beta)) = 0$ that $R_1 = 0$. The general solution of the differential equation for b_1, which is obtained by equating R_1 to zero, has the form

$$b_1 = c_1\left(\beta + \sqrt{\beta^2 + c_2^2}\right)^2$$

where c_1, c_2 are arbitrary constants. When account is taken of the condition $b_1(0) = 0$, it follows from this that $b_1 = O(\beta^2)$.

When $\beta = -\pi/2 + t$, $t \to 0$, we can represent $\psi(\varkappa, \beta)$ in the form

$$\psi(\varkappa,\beta)=1+K(\varkappa)\delta_1(t)+O(\delta_2(t))\quad (\delta_2(t)/\delta_1(t)\to 0) \tag{4.3}$$

According to relations (4.3) and (3.2)

$$\psi_\tau=-\frac{3}{2}K(\varkappa)\varkappa^{-2/3}t^{1/3}\delta_1'+O(t^{4/3}\delta_1,t^{1/3}\delta_2') \tag{4.4}$$

By expressions (1.4) and (4.4)

$$\frac{1}{2}r^2\Big|_{BC} = 1+\int_0^{\theta} \psi_\tau\Big|_{BC} \sin\theta d\theta$$

$$\psi_\tau|_{BC} = \lim_{t\to 0}\left\{-\frac{3}{2}K(|\mu\theta|)|\mu\theta|^{-2/3} t^{1/3}\delta_1' + O(t^{4/3}\delta_1, t^{1/3}\delta_2')\right\}$$

In a solution of the problem exists, then $r|_{BC}$ is a finite function of θ, which is not identically equal to unity when $\theta_0 \neq 0$. This is only possible when $\delta_1 = O(t^{2/3})$.

5. ANALYSIS OF THE FORM OF THE JET WHEN $M_c = 1$

Using relations (1.2) and the properties of the solution of the problem which have been established, we shall now show that, when $M_c = 1$, the velocity in the jet is evened out at a finite distance from the edge of the nozzle and that the surface, in which this evening-out occurs, is a surface perpendicular to the x axis. In estimating any function $\Omega(\varkappa, \beta)$, we shall use the notation $\Omega = O(\varkappa^k, \beta^m, t^n)$ which means that

$$\Omega = \begin{cases} O(\varkappa^k) & \text{when } \varkappa\to 0, \beta\in[-\pi/2,0] \\ O(\beta^m) & \text{when } \beta\to 0,\ \varkappa\leqslant\varkappa_m=(1+\mu^2\theta_0^2)^{1/2} \\ O(t^n) & \text{when } \beta=-\pi/2+t,\ t\to 0,\ \varkappa\leqslant\varkappa_m \end{cases}$$

It was established above that, when $M_c = 1$

$$\psi=\psi(\varkappa,\beta)=\psi_1+A,\quad \psi_1=g_1(\beta),\quad A=O(\varkappa^{2/3},\beta^2,t^{2/3}) \tag{5.1}$$

It follows from this that

$$\begin{aligned}
&\psi_\tau=\psi_{1\tau}+B,\quad \psi_{1\tau}=\tfrac{3}{2}\varkappa^{-2/3}\sin\beta(\cos\beta)^{1/3}g_1',\quad B=O(\varkappa^0,\beta^2,t^0)\\
&\psi_\theta=\psi_{1\theta}+O(\varkappa^{-1/3},\beta,t^{2/3}),\quad \psi_{1\theta}=\mu\varkappa^{-1}\cos\beta g_1'\\
&(1-M^2)\psi_\theta^2=\frac{9}{4}\varkappa^{-4/3}(\cos\beta)^{8/3}g_1'^2+O(\varkappa^{-2/3},\beta^2,t^2)\\
&\tau^2\psi_\tau^2=\frac{9}{4}\varkappa^{-4/3}\sin^2\beta(\cos\beta)^{2/3}g_1'^2+O(\varkappa^{-2/3},\beta^4,t^0)\\
&P\operatorname{ctg}\theta=\frac{9}{4}\mu^{-1}\varkappa^{-1/3}\sin\beta(\cos\beta)^{2/3}g_1'^2+O(\varkappa^{1/3},\beta^3,t^0)
\end{aligned} \tag{5.2}$$

In integrals of the type (1.4)

$$\zeta=\text{const},\quad d\theta=\mu^{-1}|\zeta|^{1/2}d\beta/\cos^2\beta$$

Hence

$$\begin{aligned}
&\int_0^\theta \psi_{1\tau}\theta d\theta=\frac{3}{2}\mu^{-2}\varkappa^{4/3}(\cos\beta)^{4/3}\int_0^\beta g_1'\frac{\sin^2\beta}{\cos^2\beta}d\beta=O(\varkappa^{4/3},\beta^4,t^0)\\
&\int_0^\theta \psi_1\theta d\theta=\mu^{-2}\varkappa^2\cos^2\beta\int_0^\beta g_1\frac{\sin\beta}{\cos^3\beta}d\beta=O(\varkappa^2,\beta^4,t^0)
\end{aligned} \tag{5.3}$$

Suppose $|A|$, $|B| \leqslant B_0, x \leqslant x_m$ when $-\pi/2 \leqslant \beta \leqslant 0$. Then

$$\left|\int_0^\theta \begin{Bmatrix} A \\ B\end{Bmatrix}\sin\theta d\theta\right| \leqslant \frac{1}{2}B_0\mu^{-2}\varkappa^2\sin^2\beta \tag{5.4}$$

Taking account of relations (5.1)–(5.4), we obtain from (1.4) that

$$Y = g_1 + O(\varkappa^{2/3}, \beta^2, t^0)$$
$$S = C_0 + C_1, \quad C_0 = 2g_1 + \sin\beta\cos\beta g_1', \quad C_1 = O(\varkappa^{2/3}, \beta^2, t^0) \tag{5.5}$$

In the interval $[-\pi/2, 0]$ $C_0 > 0$, we have that $\beta \to 0$ as $C_0 = O(\beta^2)$. Hence, $|C_1/C_0| < \infty$ and we can write

$$S = C_0(1 + O(\varkappa^{2/3}, \beta^0, t^0)) \tag{5.6}$$

It follows from relations (5.2) and (5.6) that

$$\frac{P}{S}\operatorname{ctg}\theta = \frac{9}{4}\mu^{-1}\varkappa^{-1/3}G_{12}g_1'^2 + O(\varkappa^{1/3}, \beta, t^0), \quad G_{mn} = \frac{\sin^m\beta(\cos\beta)^{n/3}}{2g_1 + \sin\beta\cos\beta g_1'}$$

Estimating the other terms on the right-hand sides of expressions (1.2) in a similar manner, we shall have

$$x_\tau = \frac{1}{r}\left\{\frac{9}{4}\mu^{-1}\varkappa^{-1/3}[G_{12}g_1'^2 - (\cos\beta)^{5/3}g_1'] + O(\varkappa^{1/3}, \beta, t^0)\right\} \tag{5.7}$$
$$x_\theta = \frac{1}{r}\left\{\frac{3}{2}\varkappa^{-2/3}\sin\beta(\cos\beta)^{1/3}g_1' + O(\varkappa^0, \beta^2, t^0)\right\}$$

Taking account of relations (3.2) and the equalities

$$x_\beta = x_\tau\zeta_\beta + x_\theta\theta_\beta, \quad x_\varkappa = x_\tau\zeta_\varkappa + x_\theta\theta_\varkappa$$

we find from (5.7) that

$$x_\beta = \frac{1}{r}\left\{\frac{3}{2}\mu^{-1}\varkappa^{1/3}G_{21}g_1'^2 + O(\varkappa, \beta^2, t^{-1/3})\right\}$$
$$x_\varkappa = \frac{1}{r}\left\{3\mu^{-1}\varkappa^{-2/3}G_{01}g_1g_1' + O(\varkappa^0, \beta, t^0)\right\} \tag{5.8}$$

In accordance with (5.5), $r^2 = 2g_1 + O(\varkappa^{2/3}, \beta^2, t^0)$. Since $g_1 > 0$ when $\beta \in [-\pi/2, 0]$ and $g_1 = O(\beta^2)$ as $\beta \to 0$, then

$$r^{-1} = (2g_1)^{-1/2}(1 + O(\varkappa^{2/3}, \beta^0, t^0)) \tag{5.9}$$

According to relations (5.7) – (5.9)

$$x_\theta = \frac{3\sqrt{2}}{4}\varkappa^{-2/3}\sin\beta(\cos\beta)^{1/3}g_1'g_1^{-1/2} + O(\varkappa^0, \beta, t^0)$$
$$x_\varkappa = \frac{3\sqrt{2}}{2}\mu^{-1}\varkappa^{-2/3}G_{01}g_1'g_1^{1/2} + O(\varkappa^0, \beta^0, t^0) \tag{5.10}$$
$$x_\beta = \frac{3\sqrt{2}}{4}\mu^{-1}\varkappa^{1/3}G_{21}g_1'^2g_1^{-1/2} + O(\varkappa, \beta, t^{-1/3})$$

Since $\varkappa = -\mu\theta$ when $\beta = -\pi/2$, then, when account is taken of (3.7), from the first two equalities of (5.10) we simultaneously obtain

$$x_\theta\big|_{BC} = \frac{\sqrt{2}}{2}\mu^{-2/3}p|\theta|^{-2/3} + D, \quad |D| < \infty \tag{5.11}$$

It follows from (5.11) that, when $M_c = 1$, the projection of an arc of the free surface bc on the x axis is a finite quantity (c is the point at which the curvilinear segment of the arc of the free surface joins the linear segment).

We now consider the expression

$$J = J_1 + J_2 + J_3; \quad J_1 = \int_{-\theta_0}^{-\theta_1} x_\theta d\theta, \quad J_2 = \int_{-\pi/2}^{\beta_1} x_\beta d\beta, \quad J_3 = \int_{\varkappa_1}^{0} x_\varkappa d\varkappa$$
$$\theta_1 \in (0, \theta_0), \quad \varkappa_1 = \mu |\theta_1|, \quad \beta_1 \in [-\pi/2, 0]$$

The integral J_1 is calculated when $\beta = -\pi/2$ (along BC), J_2 is calculated when $\varkappa = \varkappa_1$ and J_3 is calculated when $\beta = \beta_1$. It is obvious that J is the projection on the x axis of an arc which joins the edge of the nozzle b to the point on the streamline $\psi = g_1(\beta_1)$ at which evening-out of the velocity occurs (at which M becomes unity and θ vanishes). It follows from relations (5.10) that $J_2, J_3 = O(\varkappa_1^{1/3})$ when $\varkappa_1 \to 0$. Letting $\varkappa_1$ tend to zero, we can show that J is independent of β_1 and, consequently, the values $M = 1$, and $\theta = 0$ are attained at one and the same value of x for all the streamlines in the jet. Beyond this equalization plane, the gas velocity is equal to the velocity of sound and the jet has the shape of a cylinder.

Hence, the assertion formulated at the beginning of this section is proved. Note that, in constructing the functions ψ_1 and ψ_2, we have only used conditions in the free surface ($\beta = -\pi/2$) and on the x axis ($\beta = 0$). Hence, assuming that a solution of the problem exists for an axially symmetric nozzle of arbitrary shape, the result also holds for an arbitrary analytic dependence of the Mach number on the reduced velocity.

A similar result for a plane symmetric jet of a perfect gas, flowing from a vessel with straight walls, was obtained for the first time in [5]. Extension to the case of a plane jet of gas flowing from a vessel of arbitrary shape in the case of an arbitrary relation between the density and pressure can be found in [6].

Using the expression $\psi_1 = f_1(\omega)$ which has been found above, it can be shown that, when $M_c < 1$, evening-out of the velocity in the jet occurs at a finite distance from the nozzle edge.

The problem of the axial by symmetric emission of a gas jet from a nozzle with a curvilinear wall has been investigated using the methods of functional analysis in [7–10]. The nozzle shape was specified by the equation $r = f(x)$ $(-\infty < x \leq 0)$. Subject to certain constraints on the gas dynamic functions and the conditions

$$f(x) \in C^4, f''(x) \leq 0, \quad |\operatorname{arctg} f'(x)| < \pi/2$$
$$f(x) \equiv \text{const when } |x| > X \quad (X = \text{const} > 0)$$

the solvability of the problem was proved and it was established that equalization of the velocity in the jet with a critical pressure on the free boundary occurs at a finite distance from the nozzle edge (the shape of the surface on which equalization occurs was not investigated).

6. THE CALCULATION SCHEME

We put

$$\psi^0 = \left[(\varphi - 1)\cos\frac{\pi\theta}{2\theta_0} + 1\right]\sin^2\frac{\pi\tau}{2} + \frac{1-\cos\theta}{1-\cos\theta_0}\cos^2\frac{\pi\tau}{2}$$
$$\varphi = f_1(\omega) \text{ when } M_c < 1 \tag{6.1}$$
$$\varphi = g_1(\beta) + \varkappa^{2/3}(\varphi_1(\beta) + \delta\varphi_2(\beta))\exp(-\alpha_1\theta^2), \quad \alpha_1 \approx 10 \text{ when } M_c = 1$$

The function ψ^0, constructed in this manner, satisfies boundary conditions (1.3) and retains the same leading parts of the asymptotic expansions of the function ψ which have been found when $M_c < 1$ and $M_c = 1$.

The function $\chi = \psi - \psi^0$ must serve as a solution of the boundary-value problem

$$L(\chi) = N(\psi^0 + \chi) - L(\psi^0), \quad \chi = 0 \text{ on } AA_1BC$$

(see (2.5)). The determination of the function χ reduces to solving an iterative sequence of linear difference boundary-value problems, and the $(n+1)$-th approximation of the required function $\chi^{(n+1)}$

is found using the scheme

$$\chi^{(n+1)} = (1-w)\chi^{(n)} + w\chi^{(n+\frac{1}{2})}, \quad 0<w\leqslant 1, \quad n=0,1,\ldots$$

where the difference solution of the problem for the equation

$$L(\chi) = N(\psi^0 + \chi^{(n)}) - L(\psi^0).$$

is taken for $\chi^{(n+1/2)}$

A finite difference scheme with a five-point approximation on a uniform rectangular mesh is used in the domain Σ. The method of successive upper relaxation is used for its implementation. The transition into the physical plane is made using formulae (1.2) with a spline approximation of the mesh values of ψ.

During the course of the iterative process, a domain usually arises in the neighbourhood of the segment AA_1 where the values of the quantities $\psi^{(n)} = \psi^0 + \chi^{(n)}$, $Y^{(n)} = Y(\psi^{(n)})$, $S^{(n)} = 2Y^{(n)} + \psi_\theta^{(n)}\sin\theta$ are negative, but these values subsequently become smaller in magnitude and vanish. The following technique is used in order that the expression $N(\psi^{(n)})$ should not become infinite and that the iterative process should not diverge. When $S^{(n)} = -m^{(n)} < 0$ $S^{(n)}$, $S^{(n)} + 4m^{(n)}f(\tau,\theta)$ is replaced by $f(\tau,\theta)$, where $(\tau,\theta) \to (1,0)$ is a smooth positive function which vanishes when $\exp(-\alpha_1\theta^2)$ and rapidly tends to 1 on moving away from point C. The introduction of the factor in expression (6.1) serves similar surfaces.

7. RESULTS OF THE CALCULATIONS

Calculations were carried out on the emission of a jet of perfect gas with an adiabatic exponent $\gamma = 1.4$ for the series of values $\theta_0 \in [7,5°; 180°]$ and $M_c \in [0; 1]$. A $M_c < 1$ mesh was used when $I \times J = 200 \times 100$ and a $M_c = 1$ mesh when $I \times J = 100 \times 200$ (I and J are the number of steps along the τ and θ axes). Note that $u_1 = -(1+\gamma)$, $\delta = -u_2\mu^2 = 9(2\gamma-1)/8$ in the case of a perfect gas.

Suppose r_b, r_c are the values of r at point b and c and that $k_a = r_c^2/r_b^2$ is the jet contraction factor. The values of k_c found are shown in Table 1. The last row in this table contains the exact values of k_a, determined for $\theta_0 = 180°$ using a momentum theorem [3].

$$k_a = (\gamma M_c^2)^{-1}\{[1+(\gamma-1)M_c^2/2]^{\gamma/(\gamma-1)} - 1\} \tag{7.1}$$

The condition a_1b is satisfied with a sufficiently high accuracy for all versus of the calculation on $(r-r_b)\cos\theta_0 = -x\sin\theta_0$. It would be expected that the error in determining θ_0 would become smaller as θ_0 decreases.

Suppose k_p is the jet contraction factor in plane flow, similar to that considered above. A table of values of k_p, calculated with a high accuracy, has been presented in [11]. Comparison shows that $k_p/k_a > 1$ for all $\theta_0 \neq 0, 180°$ (when $\theta_0 + 180°$, k_p, like k_a, is determined using formula (7.1)). The ratio k_p/k_a reaches maximum values in the neighbourhood of $\theta_0 = 60°$ and, when $\theta_0 = 60°$ $k_p/k_a = 1.0351$ for $M_c = 0$ and $k_p/k_a = 1.0257$ for $M_c = 1$.

The values of x_c/r_b when $M_c = 1$ are presented below for a number of values of θ_0 (x_c is the abscissa

Table 1

θ_0°	$M_c^2 = 0$	0.2	0.4	0.6	0.8	1
7.5	0.93804	0.94379	0.95063	0.95857	0.96761	0.97753
15	0.88305	0.89375	0.90541	0.91823	0.93250	0.94862
30	0.79323	0.81007	0.82809	0.84741	0.86819	0.89067
45	0.72339	0.74385	0.76551	0.78846	0.81276	0.83863
60	0.66864	0.69124	0.71503	0.74004	0.76632	0.79409
90	0.59146	0.61608	0.64181	0.66868	0.69667	0.72608
120	0.54375	0.56903	0.59537	0.62278	0.65123	0.68104
150	0.51539	0.54084	0.56732	0.59482	0.62333	0.65315
180	0.50015	0.52563	0.55211	0.57959	0.60806	0.63780
180	0.5	0.52550	0.55202	0.57957	0.60816	0.63781

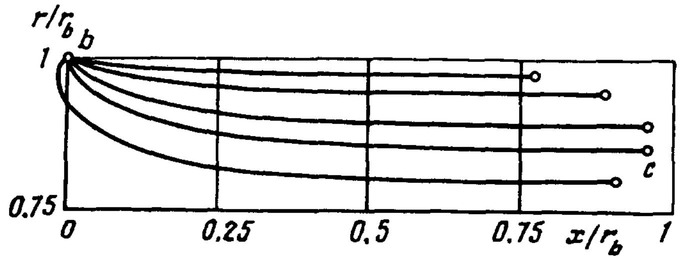

Fig. 3.

of the point c).

θ_0°	7.5	15	22.5	30	45	60	75
x_c / r_b	0.6235	0.7447	0.8124	0.8547	0.9004	0.9188	0.9233
θ_0°	90	105	120	135	150	165	180
x_c / r_b	0.9204	0.9136	0.9050	0.8959	0.8870	0.8790	0.8720

The maximum of x_c / r_b is in the neighbourhood of $\theta_0 = 75°$. When $M_c = 1$, the shape of the arc bc is shown for $\theta_0 = 15°, 30°, 60°, 90°, 180°$ in Fig. 3 (r_c decreases as θ_0 increases).

Suppose s is the arc abscissa of the curve bc. It follows from (1.2) and (1.4) that on bc.

$$\frac{r}{r_c} = \left(1 + \int_0^\theta \psi_\tau|_{BC} \sin\theta d\theta\right)^{1/2}, \quad r_c^2 = 2$$

$$\frac{1}{r_b}\frac{ds}{d\theta} = \sqrt{k_a}U, \quad U = \frac{1}{2}\psi_\tau|_{BC}\left(1 + \int_0^\theta \psi_\tau|_{BC} \sin\theta d\theta\right)^{-1/2} \tag{7.2}$$

$$\frac{x}{r_b} = \sqrt{k_a} \int_{-\theta_0}^{\theta} U\cos\theta d\theta, \quad \frac{r}{r_b} = 1 + \sqrt{k_a} \int_{-\theta_0}^{\theta} U \sin\theta d\theta$$

The values of U obtained by solving the problem at mesh points in the interval BC when $M_c = 1$ can be approximated as follows:

$$U = U_0\left(\sum_{k=1}^{6} a_k \sin k\pi t + 1 + \theta/\theta_0\right), \quad \theta \in [-\theta_0, 0]$$
$$U_0 = \frac{1}{2}\psi_{1\tau}|_{BC} = \frac{1}{2} p\mu^{-2/3} |\theta|^{-1/3}, \quad t = |\theta/\theta_0|^{0.54} \tag{7.3}$$

Table 2

θ_0^0	$a_1 \times 10^5$	$a_2 \times 10^5$	$a_3 \times 10^5$	$a_4 \times 10^5$	$a_5 \times 10^5$	$a_6 \times 10^5$
7.5	−7862	−1877	309	−307	121	−116
15	−12429	−2049	158	−304	102	−98
30	−19049	−2404	−88	−311	72	−77
45	−24039	−2767	−303	−332	42	−68
60	−28099	−3133	−502	−358	14	−61
90	−34509	−3861	−876	−422	−41	−56
120	−39475	−4568	−1234	−496	−99	−59
150	−43513	−5239	−1583	−577	−160	−67
180	−46919	−5861	−1927	−662	−225	−77

Using formulae (7.2) and (7.3) and Table 2, which contains the coefficients a_k for a number of values of θ_0, it is possible to establish the shape of the arc bc when $M_c = 1$. Here, the maximum error in determining x_c / r_b and r_c / r_b for the tabulated values of θ_0 does not exceed 0.04% and 0.003% respectively. The use of the coefficients a_k given in Table 2, obtained for intermediate values of θ_0 using the spline approximation, barely increases this error. The arc a_1bc found by the method described can serve as the generatrix of the subsonic part of an axially symmetric Laval nozzle with a plane transition surface.

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